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ON A DUCK AND ITS WINDING NUMBER IN THE MINIMAL SYSTEM

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ABSTRACT. In the generalized BVP system, its local model can not be induced well. By choosing an adequate regular transformation, which contains a new parameter u, it can be proved that this local model becomes well induced after that. Then, the winding number for a duck solution tends to infinity as a regular limit for the new parameter. In case $u-1$ is fixed sufficiently small, the number becomes large as the value of a co-parameter b embedded originally tends to 0_+, or -1_-.

1.INTRODUCTION.

The modified Bonhoeffer-van der Pol(BVP) equations were proposed by H.Kawakami et al.[5] in 1999. Their results of a computor simulation for this system show that there exist winding orbits on some projected phase space. Furthermore, the winding number increases when some parameter contained originally in this system decreases.

The BVP equations are described as follows:

$$\begin{aligned} L_1 di_1/dt &= E_1 - R_1 i_1 - v, \\ L_2 di_2/dt &= E_2 - R_2 i_2 - v, \\ C dv/dt &= i_1 + i_2 + \rho(v), \end{aligned} \tag{1.1}$$

where i_1, i_2 are the currents through the inductors L_1, L_2 and the registors R_1, R_2, respectively. Moreover, E_1, E_2 are the constant voltages, v represents at the non-linear registor ρ ($\rho(v) = v - v^3/3$) and C is a capacitor with very small capacitance. Let consider the following generalized BVP system:

$$\begin{aligned} dx/dt &= c_0 - ax - az, \\ dy/dt &= -by - bz, \\ \epsilon dz/dt &= x + y + z - z^3/3, \end{aligned} \tag{1.2}$$

where ϵ is infinitesimally small. In other words, in this paper, we use non-standard analysis by Nelson[8]. In the equation (1.1), put $i_1 = x$, $i_2 = y$, $v = z$, $C = \epsilon$ and then assume that $E_1/L_1 = c_0$(some constant),$E_2 = 0$ (for simplicity), $R_1 = R_2 = 1$, $1/L_1 = a$, $1/L_2 = b$.

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As the generalized BVP system satisfies the generic conditions; A1, ... ,A5(see Section2), some orbit expresses a jumping state with delay (duck solution, or simply duck). It may have a limit cycle containing a duck (proper duck, or duck with head) in the system. This system seems to have minimal order of the constrained surface when ϵ equals to zero. The "minimal" means for the existence of the proper duck. So, the generalized BVP system is sometimes called *a minimal system* . By giving a relation between two parameters like as $a-b=1$, this system can be reduced to the problem of one parameter family with incomplete ducks. See Section3.

In the generalized BVP system, the local model could not be induced well by itself. By introducing a certain regular transformation, which contains a parasitic parameter u, it turns to be well induced. Furthermore, it becomes clear that the winding number for the explicit duck tends to infinity (i.e.,an incomplete duck exists) as the parameter b embedded originally in the system tends to 0 near the singular value of u ($u \simeq 1$). To be brief, there exists a regular coordinate transformation containing some parameter to realize them in the minimal system. In this paper, under the above assumptions, we will provide the following two theorems and one corollary.

Theorem1. In the case $c_0 = 0$, if the parameter b satisfies $-1 < b < 0$, then the pseudo singular points are saddle, that is, the generalized BVP system has ducks. If b satisfies $-1/2 - \sqrt{8/5}/2 < b < -1, 0 < b < -1/2 + \sqrt{8/5}/2$, then the pseudo singular points are node.

Theorem2. In the case $c_0 \neq 0$ (some constant),if the parameter b giving a node point satisfies $b = O(\epsilon^2)$ when the co-parameter u holds $u - 1 = \epsilon$, then it can be obtained that there exist explicit incomplete ducks for the local model in the system.

Corollary. In the case $c_0 = 0$, if the parameter b satisfies $b = O(\epsilon^2)$ or $b = -1 - O(\epsilon^2)$ under the same conditions in the Theorem2, there exist approximately explicit ducks for the local model of the system.

2.PRELIMINARIES

Let consider a constrained system:

$$
\begin{aligned}
dx/dt &= f(x,y,z,u), \\
dy/dt &= g(x,y,z,u), \\
h(x,y,z,u) &= 0,
\end{aligned}
\tag{2.1}
$$

where u is a parameter (any fixed) and f,g,h are defined on $R^3 \times R^1$. Furthermore, let consider the singular perturbation problem of the system (2.1):

$$
\begin{aligned}
dx/dt &= f(x,y,z,u), \\
dy/dt &= g(x,y,z,u), \\
\epsilon dz/dt &= h(x,y,z,u),
\end{aligned}
\tag{2.2}
$$

where ϵ is infinitesimally small.

We assume that these systems satisfy the following conditions $(A1)-(A5)$:

$(A1)$ f and g are of class C^1 and h is of class C^2.

$(A2)$ The set $S=\{(x,y,z)\in R^3 : h(x,y,z,u)=0\}$ is a 2-dimensional differentiable manifold and the set S intersects the set $T=\{(x,y,z)\in R^3 : \partial h(x,y,z,u)/\partial z=0\}$ transversely so that the set $PL=\{(x,y,z)\in S\cap T\}$ is a 1-dimensional differentiable manifold.

$(A3)$ Either the value of f or that of g is nonzero at any point $p\in PL$.

Let $(x(t,u),y(t,u),z(t,u))$ be a solution of the system(2.1). By differentiating $h(x,y,z,u)$ with respect to the time t, the following equation holds:

$$(2.3)\quad h_x(x,y,z,u)f(x,y,z,u)+h_y(x,y,z,u)g(x,y,z,u)+h_z(x,y,z,u)dz/dt=0,$$

where $h_i(x,y,z,u)=\partial h(x,y,z,u)/\partial i$, $i=x,y,z$. The above system(2.1) becomes the following system:

$$(2.4)\quad \begin{aligned} dx/dt &= f(x,y,z,u),\\ dy/dt &= g(x,y,z,u),\\ dz/dt &= -\{h_x(x,y,z,u)f(x,y,z,u)+\\ & h_y(x,y,z,u)g(x,y,z,u)\}/h_z(x,y,z,u), \end{aligned}$$

where $(x,y,z)\in S\setminus PL$. The system(2.1) coincides with the system(2.4) at any point $p\in S\setminus PL$. In order to study the system(2.4), let consider the following system:

$$(2.5)\quad \begin{aligned} dx/dt &= -h_z(x,y,z,u)f(x,y,z,u),\\ dy/dt &= -h_z(x,y,z,u)g(x,y,z,u),\\ dz/dt &= h_x(x,y,z,u)f(x,y,z,u)+h_y(x,y,z,u)g(x,y,z,u). \end{aligned}$$

As the system(2.5) has well posedness at any point of R^3, it has well posedness indeed at any point of PL. The solutions of the system(2.4) coincide with those of the system(2.1) on $S\setminus PL$ except the velocity when they start from the same initial points.

$(A4)$ For any $(x,y,z)\in S$, the implicit function theorem holds;

$$(2.6)\quad h_y(x,y,z,u)\neq 0, h_x(x,y,z,u)\neq 0,$$

that is, the surface S can be expressed by using $y=\varphi(x,z,u)$ or $x=\psi(y,z,u)$ in the neighborhood of PL. Let $y=\varphi(x,z,u)$ exist, then the projected system, which restricts the system(2.5) is obtained:

$$(2.7)\quad \begin{aligned} dx/dt &= -h_z(x,\varphi(x,z,u),z,u)f(x,\varphi(x,z,u),z,u),\\ dz/dt &= h_x(x,\varphi(x,z,u),z,u)f(x,\varphi(x,z,u),z,u)+\\ & h_y(x,\varphi(x,z,u),z,u)g(x,\varphi(x,\varphi(x,z,u),z,u). \end{aligned}$$

$(A5)$ All the singular points of the system(2.7) are nondegenerate, the matrix induced by linearizing the system(2.7) at a singular point has two nonzero eigenvalues. Note that all the points contained in $PS=\{(x,y,z)\in PL : dz/dt=0\}$, which is called *pseudo singular points* are the singular points of the system(2.5).

Definition2.1. Let $p \in PS$ and $\mu_1(u)$, $\mu_2(u)$ be two eigenvalues of the matrix associated with the linearized system of (2.7) at p. The point p is called *pseudo singular saddle* if $\mu_1(u) < 0 < \mu_2(u)$ and called *pseudo singular node* if $\mu_1(u) < \mu_2(u) < 0$ or $\mu_1(u) > \mu_2(u) > 0$.

Definition2.2. A solution $(x(t,u), y(t,u), z(t,u))$ of the system(2.2) is called *a duck*, if there exist standard $t_1 < t_0 < t_2$ such that
(1) $^*(x(t_0,u), y(t_0,u), z(t_0,u)) \in S$, where $^*(X)$ denotes the standard part of X,
(2) for $t \in (t_1, t_0)$ the segment of the trajectory $(x(t,u), y(t,u), z(t,u))$ is infinitesimally close to the attracting part of the slow curves,
(3) for $t \in (t_0, t_2)$, it is infinitesimally close to the repelling part of the slow curves, and
(4) the attracting and repelling parts of the trajectory are not infinitesimally small.
If a duck exists as a part of a limit cycle, it is called *a proper duck.*

Definition2.3. Let E be a set in R^3. We call a point p is a δ- micro-galaxy of E when the distance from p to E is less than $exp(-n/\delta)$, where n is some positive integer and $\delta = \epsilon/\alpha^2$ (α is infinitesimally small).

Definition2.4. Let θ is an angle of the polar coordinate after changing the coordinates in the "local model" such as the orbit passing through the pseudo singular point becomes the z-axis itself. See [3],[4]. Then, the winding number $N(\psi)$ of a duck ψ is defined as follows:

$$N(\psi) = (1/2\pi)\int_{\psi} d\theta, \tag{2.8}$$

where ψ is contained partially in the δ-micro- galaxy of γ_μ.

Theorem2.1(Benoit). In the system(2.1), if the following two conditions at a pseudo singular saddle or node point;
(1) $f(O,u) \simeq h(O,u) \simeq h_y(O,u) \simeq h_z(O,u) \simeq 0$,
(2) $g(O,u) \not\simeq 0, h_x(O,u) \not\simeq 0, h_{zz}(O,u) \not\simeq 0$, where $O = (0,0,0) \in PS$,
are satisfied, the explicit duck solutions $\gamma_{\mu_i(u)}$ in the first approximation of the local model can be constructed:

$$\gamma_{\mu_i(u)}(t) = (-\mu_i(u)^2t^2 - \delta\mu_i(u), t, \mu_i(u)t)(i = 1,2), \tag{2.9}$$

wehere δ is an infinitesimally small constant.

The above Definition2.3 is based on the following fact. If ϵ is fixed arbitrarily and $\gamma(t)$ is a duck near $\gamma_{\mu(u)}(t)$ within $exp(-n/\delta)$ in some neighborhood of the pseudo singular point. See[15].

In the system(2.2), under the conditions (1) and (2) in the Theorem2.1, making the following coordinate transformations (2.10) and (2.11) successively;

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha^2\tilde{x} \\ \alpha\tilde{y} \\ \alpha\tilde{z} \end{pmatrix}, (\alpha \simeq 0, \epsilon/\alpha^2 \simeq 0) \tag{2.10}$$

(2.11)
$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} h_x(0,u)h_{zz}(0,u)\tilde{x}/2 + (h_{yy}(0,u)h_{zz}(0,u) - h_{yz}(0,u)^2)\tilde{y}^2/4 \\ \tilde{y}/g(0,u) \\ -h_{yz}(0,u)\tilde{y}/2 - h_{zz}(0,u)\tilde{z}/2 \end{pmatrix},$$

the following local model is obtained:

$$\begin{aligned} dX/dt &= pY + qZ + \xi(X,Y,Z,u), \\ dY/dt &= 1 + \eta(X,Y,Z,u), \\ \delta dZ/dt &= -(Z^2 + X) + \zeta(X,Y,Z,u), \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} p &= g(0,u)h_x(0,u)(f_y(0,u)h_{zz}(0,u) - f_z(0,u)h_{yz}(0,u))/2 \\ &+ g(0,u)^2(h_{yy}(0,u)h_{zz}(0,u) - h_{yz}(0,u)^2)/2, \\ q &= -h_x(0,u)f_z(0,u), \\ \delta &= \epsilon/\alpha^2. \end{aligned} \tag{2.13}$$

Here $\xi(X,Y,Z,u)$,$\eta(X,Y,Z,u)$ and $\zeta(X,Y,Z,u)$ are infinitesimally small when X, Y and Z are limited. Note that the solutions (2.9) are in the first approximation of the system(2.12).

By applying the following transformations of the coordinates as mentioned above, in Definition2.4, successively;

$$\begin{aligned} u &= X + Z^2 + \delta\mu, \\ v &= Y - Z/\mu, \\ z &= Z, \end{aligned} \tag{2.14}$$

$$\begin{aligned} u &= rcos\theta, \\ v &= rsin\theta, \end{aligned} \tag{2.15}$$

the Hermite equation is obtained. This equation associated with $\gamma_{\mu_i(u)}$ $(i = 1,2)$ is the following:

$$\delta\ddot{z} - \tau\dot{z} + K_i z = 0, \dot{z} = dz/d\tau, t = \tau/\alpha, (i = 1,2), \tag{2.16}$$

where K_i is a positive integer and $K_1 = 1 + \mu_2(u)/\mu_1(u)$, $K_2 = 1 + \mu_1(u)/\mu_2(u)$. See [3].

It is said that a duck $\psi(t)$ has *k jumps* if the shadow of it contains k vertical segments and that $\psi(t)$ is *long* if it lies in an infinitesimally small neighborhood at the pseudo singular point. It can be proved that if ψ is not long, the standard part of the winding number $N(\psi_i)$ associated with μ_i is an integer. If the pseudo singular point is node, it is positive. If the point is saddle, it needs a certain condition such as K_i is positive. The relation between $N(\psi_i)$ and K_i $(i = 1,2)$ is as follows.

Theorem2.2(Benoit). If the duck ψ_1, which is not long has 2 jumps, $N(\psi_1) \approx -[K_1/2]$, and if the duck ψ_2 has 2 jumps, $N(\psi_2) \approx 0$.

3. Incomplete ducks.

Definition3.1. In the system(2.12), if the following conditions (1) and (2):
(1) for any limited parameter u,
it satisfies the conditions $(A1)$-$(A5)$ and has a duck,
(2) when the parameter u tends to infinity, one of the winding numbers tends to infinity and the other tends to zero,
and the system does not have a duck as a singular limit,
are established, this solution is called an *ω-incomplete duck* .

Definition3.2. A solution $\psi(x,u)$ is called S^1 *at a* ,
if there exists a real number b such that

$$\frac{\psi(x,u)-\psi(y,u)}{x-y} \approx b, \tag{3.1}$$

for any $x, y (x \approx a, y \approx a)$.
A duck is called an S^1*duck* if it is S^1 in some neighborhood of the pseudo singular point.

Theorem3.1(Benoit). In the first approximation of the system(2.12),
if $\mu_1(u)/\mu_2(u)$ is positive (> 3) but no an integer, then all the S^1 ducks are exponentially close to one of the two explicit ducks and there exists non S^1 ducks.

Now, we assume that the coefficient q can take an unlimited number:

$$q = c_1 u + o(1), c_1 \not\simeq 0. \tag{3.2}$$

Then, putting the variable Z as

$$Z = (1/u)\tilde{Z}. \tag{3.3}$$

Then, the first approximation of the system(2.12) becomes the following:

$$\begin{aligned} &dX/dt = pY + c_1\tilde{Z},\\ &dY/dt = 1,\\ &(\delta/u)d\tilde{Z}/dt = -(\tilde{Z}^2/u^2 + X), \end{aligned} \tag{3.4}$$

where c_1 is a limited constant and $\delta/u \simeq 0$. The explicit duck solutions of the system(3.4) are

$$\gamma_{\mu_i(u)}(t) = (-\mu_i(u)^2t^2 - \delta\mu_i(u), t, u\mu_i(u)t)(i = 1, 2), \tag{3.5}$$

where $\mu_1(u)$, $\mu_2(u)$ ($\mu_1(u) > \mu_2(u)$) are the solutions of the characteristic equation:

$$2\lambda^2 + q\lambda + p = 0, \tag{3.6}$$

that is, they are the eigenvalues of the corresponding linearized system.

Theorem3.2. In the first approximation of the system(2.12), if $\mu_1(u)/\mu_2(u)$ is positive but no integer under the condition (3.2), and all the coefficients of higher order (more than 2) for u is negligible are satisfied, then this system has an ω-incomplete duck.

Corollary3.3. In the system(2.12), if the coefficient q satisfies $q = c_1u+O(1)$, that is, $q = c_1u + c_2$ where $c_1, c_2 \not\simeq 0$ and $p > 0$ or $0 > p \geq -1/32$, then there exists a finite value u_0 which makes the winding number infinite when u tends to u_0; the corresponding duck is called *incomplete* , simply. If the other coefficient p tends to zero, one of the eigenvalues in the equation(3.6) tends to zero. As the other eigenvalue keeps non zero, in this state, the winding number for a corresponding duck tends to infnity, that is, the duck is incomplete.

Remark. In this situation, the singular perturbation problem is equivalent to the following system of two parameters family with ϵ_1 and ϵ_2:

$$\begin{aligned} \epsilon_1 dX/dt &= \epsilon_1 pY + qZ + \xi(X,Y,Z,\epsilon_1,\epsilon_2), \\ dY/dt &= 1 + \eta(X,Y,Z,\epsilon_1,\epsilon_2), \\ \epsilon_2 dZ/dt &= -(Z^2 + X) + \zeta(X,Y,Z,\epsilon_1,\epsilon_2), \end{aligned} \tag{3.7}$$

where ϵ_1 and ϵ_2 are infinitesimally small.

4. The transformed BVP system

The system(1.2) could not have the local model because the third equation does contain the variable y and its coefficient is 1 (not sufficiently small). There exists a problem how to avoid this trouble in order to obtain the local model describing the explicit solution for the linearized system. This problem will be solved as follows.

Lemma4.1. *In the system*(1.2)*, there exists a regular coordinate transformation in order to induce the local model well.*

(proof)
Choosing the following transformation:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 & u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{4.1}$$

where $u(\neq 1)$ is a parasitic parameter, the system(1.2) becomes

$$\begin{aligned} dX/dt &= c_0 - aX + u(a-b)Y - (a+bu)Z, \\ dY/dt &= -bY - bZ, \\ \epsilon dZ/dt &= X + (1-u)Y + Z - Z^3/3. \end{aligned} \tag{4.2}$$

Considering the time scaled constrained system of the equation(4.2) with $a - b = 1$, the following

$$\begin{aligned} dX/dt &= (c_0 - (b+1)X + uY - (bu+b+1)Z)(1-Z^2), \\ dY/dt &= (-bY - bZ)(1-Z^2), \\ dZ/dt &= -c_0 + (b+1)X - (bu+u-b)Y + (2b+1)Z, \end{aligned} \tag{4.3}$$

is obtained, where b, u are parameters. Substituting the following

(4.4) $$Y = (X + Z - Z^3/3)/(u-1),$$

for the above equation(4.3), the equation can be projected to (X-Z) plane:

(4.5) $$\begin{aligned} dX/dt &= (c_0 - (b+1-u/(u-1))X - (bu+b+1-u/(u-1))Z \\ &\quad - uZ^3/3(u-1))(1-Z^2), \\ dZ/dt &= -c_0 + (1-u/(u-1))X + (b+1-u/(u-1))Z \\ &\quad + (b+u/(u-1))Z^3/3. \end{aligned}$$

It is important to note that there exists a singularity at $u = 1$. Then, there exist two pseudo singular points P_{0-}, P_{0+} satisfying the generic condition $(A5)$:

(4.6) $$\begin{aligned} P_{0-} &= (X_{0-}, Y_{0-}, Z_{0-}), \\ X_{0-} &= (-(4b+1)u + 4b + 3)/3 - c_0(u-1), \\ Y_{0-} &= (-(4b+1)u + 4b + 1)/3(u-1) - c_0, \\ Z_{0-} &= -1, \end{aligned}$$

and

(4.7) $$\begin{aligned} P_{0+} &= (X_{0+}, Y_{0+}, Z_{0+}), \\ X_{0+} &= ((4b+1)u - 4b - 3)/3 - c_0(u-1), \\ Y_{0+} &= ((4b+1)u - 4b - 1)/3(u-1) - c_0, \\ Z_{0+} &= 1, \end{aligned}$$

where

(4.8) $$c_0 - aX_0 + uY_0 - (a+bu)Z_0 \simeq 0, (X_0, Y_0, Z_0) \in PS.$$

The values of p, q in the equation(2.13) at these points are

(4.9) $$\begin{aligned} p_- &= -ub(Y_{0-} - 1), \\ q_- &= b(u+1) + 1, \\ p_+ &= ub(Y_{0+} + 1), \\ q_+ &= q_-. \end{aligned}$$

Since the equation(4.3) satisfies the assumptions $(1), (2)$ in Theorem2.1, it becomes clear that the local model can be induced well. This completes the proof of Lemma4.1.

Lemma4.2. *The local model of the system*(4.2) *is an approximation of this system in the neighborhood of the pseudo singular point.*

(proof)

In generally, after the changing the coordinates(2.11), the following new system is obtained:

(4.10) $$\begin{aligned} du/dt &= (dX/dt)/\alpha^2 = f(\alpha^2 u, \alpha v, \alpha w)/\alpha^2, \\ dv/dt &= (dY/dt)/\alpha = g(\alpha^2 u, \alpha v, \alpha w)/\alpha, \\ \epsilon dw/dt &= \epsilon(dZ/dt)/\alpha = h(\alpha^2 u, \alpha v, \alpha w), \\ (\epsilon/\alpha^2) dZ/dt &= h(\alpha^2 u, \alpha v, \alpha w)/\alpha^2. \end{aligned}$$

Furthermore, after the transformation(2.11), the higher terms of f/α^2 and g/α are zero in the system(4.2) ($\xi(X, Y, Z, u) = \eta(X, Y, Z, u) = 0$). The higher term of h/α^2 is negligible. In fact,

$$
\begin{aligned}
\zeta(X,Y,Z,u) &= (h_{XXX}(P_{0-})X^3 + h_{YYY}(P_{0-})Y^3 + h_{ZZZ}(P_{0-})Z^3 \\
&+ 3(h_{XXY}(P_{0-})X^2Y + h_{XXZ}(P_{0-})X^2Z + h_{YYZ}(P_{0-})Y^2Z \\
&+ h_{YZZ}(P_{0-})YZ^2 + h_{XYY}(P_{0-})XY^2 + h_{XZZ}(P_{0-})XZ^2) \\
&+ 6h_{XYZ}(P_{0-})XYZ)/3!\alpha^2 = -2\alpha w^3/3!,
\end{aligned} \tag{4.11}
$$

since $h_{ZZZ}(P_{0-}) = -2$, $Z^3 = \alpha^3 w^3$ and other partial derivatives are all zeroes. Then, by using the assumptions(1), (2) in Theorem2.1, the local model can be obtained. It is confirmed that the model is an approximation of the system(4.2) near the point P_{0-}. In the case at P_{0+}, it is confirmed also in the same way. This completes the proof of Lemma4.2.

Lemma4.3. *There exists an explicit duck in the local model for the system*(4.2) *under the condition*(4.8). *If the duck is proper, that is, if there exists a limit cycle, which has a duck as a part of the solution, the right hand side of the third equation in the system*(4.2) *has minimal order (minimal system) for the existence of the proper duck.*

(proof)
By the Theorem2.1, the explicit duck near the point P_{0-} is obtained as follows:

$$
\gamma_{\mu_i(b,u)}(t) = (-\mu_i(b,u)^2t^2 - \delta\mu_i(b,u) + X_{0-}, t + Y_{0-}, \mu_i(b,u)t + Z_{0-})(i = 1, 2), \tag{4.12}
$$

where $\mu_i(b,u)$ is an eigenvalue for the linealized system of the system(4.5). If the order of the polynomial is smaller than 3, there is no jumping orbit to return to an initial point. Thus, there is no limit cycle, which contains a duck. This completes the proof of Lemma4.3.

In this paper, the condition for the existence of the proper duck does not be treated.

5. The proofs of Theorem1 and Theorem2

In the regularized system(4.3), if $c_0 = 0$ is satisfied, there are three singular points in the time scaled reduced system:

P_{0z} ($z = -, 0, +$); $P_{0-} = (1 + 4b/3, -1/3 - 4b/3, -1)$, $P_{00} = (0, 0, 0)$, $P_{0+} = (-1 - 4b/3, -5/3 - 4b/3, 1)$. In this state, only P_{0+}, P_{0-} are pseudo singular points. It is possible to induce them by calculating the Jacobian matrix of the time scaled reduced system(4.4) directly. In fact, the constrained system for the system(4.2) becomes the following equation corresponding to the equation(2.7):

$$
\begin{aligned}
dx/dt &= -(1 + b)(x + z)(1 - z^2), \\
dz/dt &= x + (1 + b)z + bz^3/3,
\end{aligned} \tag{5.1}
$$

where only b is a parameter. Now, let consider these two points P_{0-} and P_{0+}.

In both cases, the characteristic equation associated with the linearized system of the system(5.1) are quite the same as follows:

$$\lambda^2 - (1+2b)\lambda + 8b(1+b)/3 = 0. \tag{5.2}$$

If b satisfies $-1 < b < 0$, the pseudo singular points P_{0-} and P_{0+} are saddle, that is, there exist ducks in this system. If b satisfies $-1/2 - \sqrt{8/5}/2 < b < -1, 0 < b < -1/2 + \sqrt{8/5}/2$, P_{0-} and P_{0+} are node points. This completes the proof of Theorem1.

In the case $c_0 \neq 0$, under the condition (4.8), if the parameter b satisfies $b = \epsilon^2$, the pseudo singular points P_{0-} and P_{0+} are node when the co-parameter u satisfies $u - 1 = \epsilon$. In fact, the characteristic equation of the local model is

$$2\lambda^2 + q_{+(-)}\lambda + p_{+(-)} = 0, \tag{5.3}$$

where $p_{+(-)}, q_{+(-)}$ are in the equations(4.9). Note that the existence of the local model is ensured from Lemma4.1. As $p_{+(-)} > 0$ (the pseudo singular points are node), the winding numbers are well defined. Furthermore, the value of p satisfying $O(\epsilon^2)$ holds the relation in Corollary3.3. Therefore, the transformed system has an incomplete duck. At that time, the corresponding index K in the equations(2.16) tends to infinity, so the winding number of the duck tends to infinity by Theorem2.2. In fact, let the parameter u is fixed very near 1, then the winding number tends to infinity as the co-parameter b tends to 0_{+0}. In this state, one of the eigenvalues tends to zero and the other one keeps nonzero since $q_+ = q_-$ tends to 1. Thus, this system has an incomplete duck from Corollary3.3. It should be also available if u holds $u - 1 = O(\epsilon)$ whenever b satisfies $b = O(\epsilon^2)$. This completes the proof of Theorem2.

Remark1. By using an affine transformation:

$$x = X - c_0/a, y = Y, z = Z, \tag{5.4}$$

we can reduce the system(1.2) to the following:

$$\begin{aligned} dx/dt &= -ax - az, \\ dy/dt &= -by - bz, \\ \epsilon dz/dt &= c_0/a + x + y + z - z^3/3. \end{aligned} \tag{5.5}$$

Then, the time scaled reduced sysytem associated with the system(2.5) is

$$\begin{aligned} dx/dt &= (-ax - az)(1 - z^2), \\ dy/dt &= (bx - bz^3/3 + bc_0/a)(1 - z^2), \\ dz/dt &= -bc_0/a + x + az + bz^3/3. \end{aligned} \tag{5.6}$$

Since the parameter b tends to zero (the parameter a tends to 1), bc_0/a tends to zero, the explicit duck near the pseudo singular point P_{0+} in the local model tends to the following:

$$\begin{aligned} &\gamma_{\mu_i(b,u)+}(t) = \\ &(\mu_i^2(b,u)t^2 - \delta\mu_i(b,u) - X_{0+} - (aX_{0+} - Y_{0+} + (b+2)Z_{0+})/a, \\ &t - Y_{0+}, \mu_i(b,u)t - Z_{0+})(i = 1,2). \end{aligned} \tag{5.7}$$

In the above limit, the relation $c_0 = 0$ holds. When it is near on the point P_{0-}, the explicit duck is close to $\gamma_{\lambda_i(b,u)-}$:

$$\begin{aligned} &\gamma_{\lambda_i(b,u)-}(t) = \\ &(-\lambda_i^2(b,u)t^2 - \delta\lambda_i(b,u) + X_{0-} - (aX_{0-} - Y_{0-} + (b+2)Z_{0-})/a, \\ &t + Y_{0-}, \lambda_i(b,u)t + Z_{0-})(i = 1,2). \end{aligned} \tag{5.8}$$

Note that the condition (4.8) restricts the system(5.5).

Remark2. When the parameter b tends to $-1(b = -1 - O(\epsilon^2)$ and $a = O(\epsilon))$, the value of $u - 1$ can keep small $(u = 1 + O(\epsilon))$. It is possible to hold $c_0 = O(\epsilon^2)$ in this state. Therefore, the explicit ducks (5.8) are also available for this case. This fact coinsides with the results of the simulation [5]: when $1/L_1$ becomes smaller, the winding number becomes larger.

This completes the proof of Corollary.

Remark3. In the Bonhoeffer van der Pol system, the parameter b is one of the two inductances. It takes usually positive number. Though we assume that $a - b = 1$ first, it might be natural physically to put $a + b = 1$ and $a > 0, b > 0$ in the circuit. In this state, we can get the same result as b tends to 1.

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References

1. V.I.Arnold, *Dynamical Systems V*, Springer-Verlag (1994).
2. E.Benoit, *Systems lents-rapides dans R^3 et leurs canards,*, Asterisque **109–110** (1983), 55-71.
3. E.Benoit, *Canards et enlacements*, Publ. Math. IHES **72** (1990), 63–91.
4. N.Kakiuchi, and K.Tchizawa, *On an explicit duck solution and delay in the FitzHugh-Nagumo equation*, J. of Differential Equations **141** (1997), 327–339.
5. H.Kawakami, K.Tsumoto, and T.Yoshinaga, *Bifurcation of a modified BVP circuit model for neurons generating rectangular waves*, IEICE Trans.Fundamentals **EOO-A:4** (1999).
6. S.Martin, *A hydrodynamic curiosity:the salt Oscillator*, Geophysical Fluid Dynamics **1** (1970), 143–160.
7. A.I.Neishtadt, *On delayed stability loss under dynamical bifurcations, I*, Differential Equations **23** (1987), 2060–2067; II, ibid **24** (1988), 226–233.
8. E.Nelson, *Internal Set Theory*, Bull. Amer. Math. Soc. **83** (1977), 1165–1198.
9. K.Tchizawa, *On an omega-incomplete duck and its application*, International J. of Applied Mathematics **2:1** (2000), 25–38.
10. K.Tchizawa, *The complete ducks not satisfying S^1*, Proceedings of 9th International Colloquium on Differential Equations, VSP, (1999), 417–422.
11. K.Yoshikawa, and T.Omochi, *Chemical sensing by a novel electrical oscillator:Polysaccharides in concanavalin A solutions*, Biochemical and Biophysical Research Communications **137:3** (1986), 978–983.
12. K.Yoshikawa, S.Maeda, and H.Kawakami, *Various osillatory regimes and bifurcations in a dynamic chemical system at interface*, Ferroelectrics **86** (1988), 281–298.
13. K.Yoshikawa, K.Fukunaga, and H.Kawakami, *A tri-phasic mode is stable when three nonlinear oscillators interact with each other*, Chemical Physics Letters **174:2** (1990), 203–207.
14. K.Yoshikawa, T.Yoshinaga, H.Kawakami, and S.Nakata, *New strategy of informational processing: utilization of nonlinear dynamics to chemical sensing*, Physica A **188** (1992), 243–250.
15. A.K.Zvonkin, and M.A.Shubin, *Non-standard analysis and singular perturbations of ordinary differential equations*, Russian Math. Surveys **39** (1984), 69–131.

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